Online Appendix to The Public Housing Allocation Problem

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May 2024*

Abstract

This document contains appendix material for Murra-Anton and Thakral (2024).

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A.1 Proof of Theorem 1

Proof of Theorem 1. We prove the case of ex-post Pareto efficiency, as the case of ex-post elimination of justified envy is proved in the main text. Consider any given strategy-proof and non-wasteful mechanism φ, along with the following setup:

- **Applicants.** $I^* = \{i_1, i_2\}$.
- **Buildings.** $B^* = \{\beta_1, \beta_2, \beta_3\}$.
- **Preferences.** $\succ^*$ is a given preference profile such that

$$i_1 : (\beta_3, 1) \succ^*_{i_1} (\beta_1, 0) \succ^*_{i_1} (\beta_2, 2) \succ^*_{i_1} i_1$$
$$i_2 : (\beta_2, 2) \succ^*_{i_2} (\beta_1, 0) \succ^*_{i_2} (\beta_3, 1) \succ^*_{i_2} i_2$$

- **Priorities.** $\triangleright^*$ is any priority profile such that $i \triangleright^* \beta$ for all $i \in I$ and all $\beta \in B^*$.
- **Arrival Distribution.** π^* is such that $\pi^*(a_0) = \frac{1}{4}$ for all $a_t \in B^* \cup \{\emptyset\}$ and $t \in \{0, 1, 2\}$, and $\pi^*(a_t = \emptyset) = 1$ for all $t \notin \{0, 1, 2\}$.
- **Realized Arrivals.** $A^* = \{a \in A(\pi^*) : a_0 = \beta_1\}$.

Consider the PHAP subset

$$\mathcal{P}^* = \{P \in \mathcal{P} : I = I^*, B = B^*, \succ = \succ^*, \triangleright = \triangleright^*, \pi = \pi^*, a \in A^*\}.$$

Given that both applicants are acceptable at both buildings and vice versa, non-wastefulness of φ implies that in any $P \in \mathcal{P}^*$, arrival a_0 must be assigned and delivered at $t = 0$. Moreover, all the PHAP in $\mathcal{P}^*$ have the same applicants, buildings, preferences, priorities, distribution over arrivals, and first arrival; they only differ in $(a_t)_{t \in \mathbb{N}}$. Consequently, the informational constraint of φ implies that $i_{\varphi(P)}(a_0) \in \{i_1, i_2\}$ must be the same for all $P \in \mathcal{P}^*$. This leads to only two possibilities: $i_{\varphi(P)}(a_0) = i_1$ for all $P \in \mathcal{P}^*$ or $i_{\varphi(P)}(a_0) = i_2$ for all $P \in \mathcal{P}^*$.

<u>Case 1:</u> $i_{\varphi(P)}(a_0) = i_1$ for all $P \in \mathcal{P}^*$. As i_2 is acceptable at both buildings and vice versa, φ's non-wastefulness implies that $i_{\varphi(P)}(a_1) = i_2$ for all $P \in \mathcal{P}^*$ such that $a_1 \neq \emptyset$. This is particularly true for any $P^*_1 \in \mathcal{P}^*$ featuring $a_1 = \beta_3$. It follows that the matching outcome $\phi(P^*_1)$ constructed by swapping the assignment order between i_1 and i_2 ex-post Pareto dominates $\varphi(P^*_1)$ because $\phi_{P^*_1}(i_2) = (\langle \beta_1, 1 \rangle, 0) \succ^*_{i_2} (\langle \beta_3, 1 \rangle, 1) = \varphi_{P^*_1}(i_2)$ and $\phi_{P^*_1}(i_1) = (\langle \beta_3, 1 \rangle, 1) \succ^*_{i_1} (\langle \beta_1, 1 \rangle, 0) = \varphi_{P^*_1}(i_1)$.

<u>Case 2:</u> $i_{\varphi(P)}(a_0) = i_2$ for all $P \in \mathcal{P}^*$. As i_1 is acceptable at both buildings and vice versa, φ's non-wastefulness implies that $i_{\varphi(P)}(a_2) = i_1$ for all $P \in \mathcal{P}^*$ such that $a_1 = \emptyset \neq a_2$. This is particularly true for any $P^*_2 \in \mathcal{P}^*$ featuring $a_1 = \emptyset$ and $a_2 = \beta_2$. It follows that the

matching outcome $\phi(P_2^*)$ constructed by swapping the assignment order between i_1 and i_2 ex-post Pareto dominates $\varphi(P_1^*)$ because $\phi_{P_2^*}(i_2) = (\langle\beta_2, 1\rangle, 2) \succ_{i_2}^* (\langle\beta_1, 1\rangle, 0) = \varphi_{P_2^*}(i_2)$ and $\phi_{P_2^*}(i_1) = (\langle\beta_1, 1\rangle, 0) \succ_{i_1}^* (\langle\beta_2, 1\rangle, 2) = \varphi_{P_2^*}(i_1)$. □

Proof of Corollary 1. Theorem 1 implies that if φ is ex-post Pareto efficient, then it cannot be non-wasteful. Therefore, if φ is ex-post Pareto efficient, there exists $P \in \mathcal{P}$ such that for some $a_t \neq \emptyset$ and $i \in I \setminus \left\{i' \in I : T_{\varphi(P)}(i') < t\right\}$, we have $a_t \succ_i i$, $i \rhd_{a_t} a_t$ and either (i') $i_{\varphi(P)}(a_t) = a_t$ or (ii') $t_{\varphi(P)}(i_{\varphi(P)}(a_t)) > t$. Note, however, that Lemma 1 implies that if $i_{\varphi(P)}(a_t) \neq a_t$, then $t_{\varphi(P)}(i_{\varphi(P)}(a_t)) = t$, namely, that if (i') fails, then (ii') fails as well. We conclude that (i') must hold. □

A.2 Proof of Theorem 2

Proof of Theorem 2. Consider k, a given natural number. We define the following PHAP subset:

Definition A.1. $\mathcal{P}(k) \subset \mathcal{P}$ is an **order-*k* PHAP test set** if

$$\mathcal{P}(k) = \{(I, B, \succ, \rhd, \pi, a) : I = I(k), B = B^*, \succ = \succ^*, \rhd = \rhd^*, \pi = \pi^*, a \in A(\pi^*)\},$$

such that

- **Applicants.** $I(k)$ is an an arbitrary set of $2k$ applicants that can be split into two size-k subsets, as in $I(k) = I_\lambda(k) \cup I_\omega(k)$, for $I_\lambda(k) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $I_\omega(k) = \{\omega_1, \omega_2, \ldots, \omega_k\}$.
- **Buildings.** B^* is an arbitrary set of 2 buildings, namely, $B^* = \{\Omega, \Lambda\}$.
- **Preferences.** $\succ^*$ is an arbitrary preference profile such that both buildings are acceptable to all applicants, $(\Omega, 2k) \succ_\omega^* (\Lambda, 0)$ for all $\omega \in I_\omega(k)$, and $(\Lambda, 2k) \succ_\lambda^* (\Omega, 0)$ for all $\lambda \in I_\lambda(k)$.
- **Priorities.** $\rhd^*$ is an arbitrary priority profile such that all applicants are acceptable to both buildings and $\lambda \rhd_\Omega \omega$ and $\omega \rhd_\Lambda \lambda$ for all $\omega \in I_\omega(k)$ and $\lambda \in I_\lambda(k)$.
- **Arrival Distribution.** π^* is an arbitrary MAP distribution over B^*.

Remark A.1. A PHAP test set is constructed by fixing a set of applicants $I(k)$, a set of buildings B^*, a preference profile $\succ^*$, a priority profile $\triangle^*$, and an arrival distribution π^* with the characteristics in Definition A.1. The only difference between the PHAPs in a given test set is the realized sequence of arrivals.

Using PHAP test sets, we establish the following lemma that directly proves our desired result:

Lemma A.1. *For any natural number k, if φ is a non-wasteful allocation mechanism and $\mathcal{P}(k)$ is a order-k PHAP test set, then there is $P^* \in \mathcal{P}(k)$ such that $E(\varphi(P^*))+F(\varphi(P^*)) \geq k$.*

Proof of Lemma A.1. We prove the result by induction.

Initial Step. Suppose that $k = 1$. Consider a non-wasteful mechanism φ and any given order-1 PHAP test set $\mathcal{P}(1)$. Consider further $\mathcal{P}_0 = \{P \in \mathcal{P}(1) : a_0 = \Omega\} \subset \mathcal{P}(1)$. Given Remark A.1 and the fact that every $P \in \mathcal{P}_0$ features the same a_0, the informational constraint of φ implies that $i_{\varphi(P)}(a_0)$ must be the same for all $P \in \mathcal{P}_0$. Moreover, given that both applicants are acceptable at both buildings and vice versa, φ's non-wastefulness implies that in any $P \in \mathcal{P}_0$, a_0 must be assigned and delivered at $t = 0$. Therefore, as we have that $I_w(1) = \{\omega_1\}$ and $I_\lambda(1) = \{\lambda_1\}$ for all $P \in \mathcal{P}_0$, there are only two possibilities: $i_{\varphi(P)}(a_0) = \omega_1$ for all $P \in \mathcal{P}_0$ or $i_{\varphi(P)}(a_0) = \lambda_1$ for all $P \in \mathcal{P}_0$.

Case 1: $i_{\varphi(P)}(a_0) = \omega_1$ for all $P \in \mathcal{P}_0$. As λ_1 is acceptable at both buildings and vice versa, φ's non-wastefulness implies that $i_{\varphi(P)}(a_1) = \lambda_1$ for all $P \in \mathcal{P}_0$. This is particularly true for a given $P_1^* \in \mathcal{P}_0$ such that $a_1 = \Omega$. It follows that λ_1 ex-post justifiably envies ω_1 in outcome $\varphi(P_1^*)$ because costly-waiting preferences imply $\varphi_{P_1^*}(\omega_1) = (\langle\Omega, 1\rangle, 0) \succ^*_{\lambda_1} (\langle\Omega, 2\rangle, 1) = \varphi_{P_1^*}(\omega_2)$, while we have $\lambda_1 \rhd^*_\Omega \omega_1$. Therefore, $F(\varphi(P_1^*)) = 1$. We have shown that there exists $P_1^* \in \mathcal{P}_0 \subset \mathcal{P}(1)$ such that $E(\varphi(P_1^*)) + F(\varphi(P_1^*)) \geq 1$.

Case 2: $i_{\varphi(P)}(a_0) = \lambda_1$ for all $P \in \mathcal{P}_0$. As ω_1 is acceptable at both buildings and vice versa, φ's non-wastefulness implies that $i_{\varphi(P)}(a_1) = \omega_1$ for all $P \in \mathcal{P}_0$. This is particularly true for a given $P_1^* \in \mathcal{P}_0$ such that $a_1 = \Lambda$. It follows that the matching outcome $\phi(P_2^*)$ constructed by swapping the assignment order between i_1 and i_2 ex-post Pareto dominates $\varphi(P_2^*)$ because $\phi_{P_2^*}(\omega_1) = (\langle\Omega, 1\rangle, 0) \succ^*_{\omega_1} (\langle\Lambda, 1\rangle, 1) = \varphi_{P_2^*}(\omega_1)$ and $\phi_{P_2^*}(\lambda_1) = (\langle\Lambda, 1\rangle, 1) \succ^*_{\lambda_1} (\langle\Omega, 1\rangle, 0) = \varphi_{P_2^*}(\lambda_1)$. Therefore, $E(\varphi(P_1^*)) = 1$. We have shown that there exists $P_2^* \in \mathcal{P}_0 \subset \mathcal{P}(1)$ such that $E(\varphi(P_1^*)) + F(\varphi(P_1^*)) \geq 1$.

Inductive step. Assume that the following inductive hypothesis is true for a given natural number $k > 1$:

$$\begin{gathered}\text{If } \varphi \text{ is a non-wasteful mechanism and } \mathcal{P}(k) \text{ is an order-}k \text{ PHAP test set,} \\ \text{then there is } P^* \in \mathcal{P}(k) \text{ such that } E(\varphi(P^*)) + F(\varphi(P^*)) \geq k\end{gathered} \tag{IH}$$

We show that Equation (IH) implies that if φ is a non-wasteful allocation mechanism and $\mathcal{P}(k+1)$ is an order-$k+1$ PHAP test set, then there is $P^* \in \mathcal{P}(k+1)$ such that $E(\varphi(P^*)) + F(\varphi(P^*)) \geq k+1$.

Consider a non-wasteful mechanism φ and any given order-$k+1$ PHAP test set $\mathcal{P}(1)$. Consider further $\mathcal{P}_0 = \{P \in \mathcal{P}(k+1) : a_0 = \Omega\} \subset \mathcal{P}(k+1)$. Given Remark A.1 and the fact that every $P \in \mathcal{P}_0$ features the same a_0, the informational constraint of φ implies that $i_{\varphi(P)}(a_0)$ must be the same for all $P \in \mathcal{P}_0$. Moreover, given that all applicants are acceptable at both buildings and vice versa, φ's non-wastefulness implies that in any $P \in \mathcal{P}_0$, a_0 must be assigned and delivered at $t = 0$. Therefore, there are only two possibilities: there is $\omega_0 \in I_\omega(k+1)$ such that $i_{\varphi(P)}(a_0) = \omega_0$ for all $P \in \mathcal{P}_0$ or there is $\lambda_0 \in I_\lambda(k+1)$ such that $i_{\varphi(P)}(a_0) = \lambda_0$ for all $P \in \mathcal{P}_0$.

Case 1: $i_{\varphi(P)}(a_0) = \omega_0$ for all $P \in \mathcal{P}_0$. This is particularly true for any $P_1^* \in \mathcal{P}_0$ such that $a_t = \Omega$ for $t \in \{1, 2, \dots, 2(k+1)-1\}$, which are delivered to an applicant upon arrival, given that every remaining applicant is acceptable at both buildings and vice versa and that φ is non-wasteful. It follows that $\beta_{\varphi(P_1^*)}(\lambda_i) = \Omega$ and $t_{\varphi(P_1^*)}(\lambda_i) > 0$ for all $\lambda_i \in I_\lambda(k+1)$. It follows that $F(\varphi(P_1^*)) = k+1$ because every $\lambda_i \in I_\lambda(k+1)$ ex-post justifiably envies ω_0 in outcome $\varphi(P_1^*)$ due to costly-waiting preferences: $\varphi_{P_1^*}(\omega_0) = (\langle \Omega, 1\rangle, 0) \succ^*_{\lambda_i} (\langle \Omega, r_{\varphi(P_1^*)}(\lambda_i)\rangle, t_{\varphi(P_1^*)}(\lambda_i)) = \varphi_{P_1^*}(\lambda_i)$ and $\lambda_i \rhd^*_\Omega \omega_0$. We have shown that there exists $P_1^* \in \mathcal{P}_0 \subset \mathcal{P}(k+1)$ such that $E(\varphi(P_1^*)) + F(\varphi(P_1^*)) \geq k+1$, and the result holds for Case 1.

Case 2: $i_{\varphi(P)}(a_0) = \lambda_0$ for all $P \in \mathcal{P}_0$. Consider further $\mathcal{P}_1 = \{P \in \mathcal{P}_0 : a_1 = \Lambda\}$. Given Remark A.1 and the fact that every $P \in \mathcal{P}_1$ features the same a_0, a_1, and that $i_{\varphi(P)}(a_0) = \lambda_0$ for all $P \in \mathcal{P}_0$, the informational constraint of φ implies that $i_{\varphi(P)}(a_1)$ must be the same for all $P \in \mathcal{P}_1$. Moreover, given that all applicants are acceptable at both buildings and vice versa, φ's non-wastefulness implies that in any $P \in \mathcal{P}_1$, a_1 must be assigned and delivered at $t = 1$. Then, there are only two possibilities: there is $\lambda_1 \in I_\lambda(k+1) \setminus \{\lambda_0\}$ such that $i_{\varphi(P)}(a_1) = \lambda_1$ for all $P \in \mathcal{P}_1$ or there is $\omega_1 \in I_\omega(k+1)$ such that $i_{\varphi(P)}(a_1) = \omega_1$ for all $P \in \mathcal{P}_1$.

- *Sub-Case 2.1:* $i_{\varphi(P)}(a_1) = \lambda_1$ for all $P \in \mathcal{P}_1$. This is particularly true for any $P_{2.1}^* \in \mathcal{P}_1$ such that $a_t = \Lambda$ for $t \in \{2, \dots, 2k+1\}$, which are delivered to an applicant upon arrival, given that every remaining applicant is acceptable at both buildings and vice versa and that φ is non-wasteful. It follows that $\beta_{\varphi(P_{2.1}^*)}(\omega_i) = \Lambda$ and $t^{\varphi}_{\varphi(P_{2.1}^*)}(\omega_i) > 1$ for all $\omega_i \in I_\Omega(k+1)$. It follows that $F(\varphi(P_{2.1}^*)) = k+1$ because every $\omega_i \in I_\lambda(k+1)$ ex-post justifiably envies λ_1 in outcome $\varphi(P_{2.1}^*)$ due to costly-waiting preferences: $\varphi_{P_{2.2}^*}(\lambda_1) = (\langle \Lambda, 1\rangle, 1) \succ^*_{\omega_i} (\langle \Lambda, r_{\varphi(P_{2.1}^*)}(\omega_i)\rangle, t_{\varphi(P_{2.1}^*)}(\omega_i)) = \varphi_{P_{2.2}^*}(\omega_i)$ and $\omega_i \rhd^*_\Lambda \lambda_1$. We have shown that there exists $P_{2.1}^* \in \mathcal{P}_1 \subset \mathcal{P}(k+1)$ such that $E(\varphi(P_{2.1}^*)) + F(\varphi(P_{2.1}^*)) \geq k+1$, and the result holds for Case 2.1.
- *Sub-Case 2.2:* $i_{\varphi(P)}(a_1) = \omega_1$ for all $P \in \mathcal{P}_1$. Given a non-wasteful mechanism φ ex-post Pareto dominated by another non-wasteful mechanism ϕ, any $P \in \mathcal{P}_1$, and any subset $I_S \subset I(k+1)$, we define

$$F(\varphi(P) \mid I_S) = \sum_{i \in I_S} |\{i' \in I_S : i' \text{ ex-post justifiably envies } i \text{ in outcome } \varphi(P)\}|,$$

$$E(\varphi(P), \phi(P) \mid I_S) = |\{i \in I_S : \phi_P(i) \succ_i \varphi_P(i)\}|.$$

$F(\varphi(P) \mid I_S)$ counts the ex-post elimination of justified envy violations of outcome $\varphi(P)$ involving only the applicants in I_S. Likewise, $E(\varphi(P), \phi(P) \mid I_S)$ counts the number of applicants in I_S that are strictly improved by the Pareto-dominant outcome $\phi(P)$. Given that $\varphi(P)$ can exhibit instances of justified envy involving $i \in I_S$ and $i' \in I \setminus I_S$, we have that

$$F(\varphi(P)) \geq F(\varphi(P) \mid I_S) + F(\varphi(P) \mid I(k+1) \setminus I_S). \tag{1}$$

Likewise, as $\phi(P) \in \mathcal{D}(\varphi(P))$,

$$E(\varphi(P)) \geq E(\varphi(P), \phi(P) \mid I_S) + E(\varphi(P), \phi(P) \mid I(k+1) \setminus I_S). \tag{2}$$

By defining $I_S = \{\lambda_0, \omega_1\}$, we use Equation (1) and Equation (2) to separately count the violations happening up to time $t = 1$ and ones in the continuation $t \geq 2$. In the current case, $i_{\varphi(P)}(a_0) = \lambda_0$ and $i_{\varphi(P)}(a_1) = \omega_1$. Moreover, for any $P \in \mathcal{P}_1$, we have that $a_0 = \Omega$ and $a_1 = \Lambda$. Then, given $P \in \mathcal{P}_1$, we define the set of dominant matching outcomes that swap the assignments of λ_0 and ω_1:

$$\mathcal{D}^*(\varphi(P)) = \{\phi(P) \in \mathcal{D}(\varphi(P)) : \phi_P(\omega_1) = \varphi_P(\lambda_0), \phi_P(\lambda_0) = \varphi_P(\omega_1)\}.$$

We note that $\mathcal{D}^*(\varphi(P)) \neq \emptyset$ because $\hat{\phi}(P) \in \mathcal{D}^*(\varphi(P))$ if $\hat{\phi}_P(i) = \varphi_P(i)$ for all $i \in I(k+1) \setminus I_S$, $\hat{\phi}_P(\omega_1) = \varphi_P(\lambda_0)$, and $\hat{\phi}_P(\lambda_0) = \varphi_P(\omega_1)$. It follows that for any $P \in \mathcal{P}_1$ and any $\phi(P) \in \mathcal{D}^*(\varphi(P))$, $\phi_P(\lambda_0) = (\langle \Lambda, 1 \rangle, 1) \succ^*_{\lambda_0} (\langle \Omega, 1 \rangle, 0) = \varphi_P(\lambda_0)$ and $\phi_P(\omega_1) = (\langle \Omega, 1 \rangle, 0) \succ^*_{\omega_1} (\langle \Lambda, 1 \rangle, 1) = \varphi_P(\omega_1)$, so we establish that

$$E(\varphi(P), \phi(P) \mid I_S) = 2 \text{ for all } P \in \mathcal{P}_1 \text{ and } \phi(P) \in \mathcal{D}^*(\varphi(P)). \tag{3}$$

The following claim, which is involved, completes the proof by asserting that we can find a PHAP in $\mathcal{P}$ such that the matching outcome of the continuation of the problem after $t = 1$, there are at least k more violations to ex-post elimination of justified envy and Pareto efficiency. Formally,

Claim A.1. *Equation* (IH) *implies that there exists* $P^*_{2.2} \in \mathcal{P}_1$ *and* $\phi(P^*_{2.2}) \in \mathcal{D}^*(\varphi(P^*_{2.2}))$, *such that*

$$E(\varphi(P^*_{2.2}), \phi(P^*_{2.2}) \mid I(k+1) \setminus I_S) + F(\varphi(P^*_{2.2}) \mid I(k+1) \setminus I_S) \geq k. \tag{4}$$

The proof of Claim A.1 is involved and not essential for the argument, thus deferred to the end of the proof. For the sake of completeness, however, we provide intuition. Equation (3) establishes that an ex-post Pareto improvement is provided by any mechanism that solves any $P \in \mathcal{P}_1$ by swapping the matching outcomes of λ_0 and ω_1. Holding fix the fact that λ_0 and ω_0 are assigned to a_0 and a_1 respectively, the set of continuation problems generated by $\mathcal{P}_1$ constitutes an order-k PHAP test set: we are left with k λ-type applicants and k ω-type applicants; moreover, the buildings, preferences, prioirities, and arrival distributions remain as defined in $\mathcal{P}(k+1)$. Then, Equation (IH) implies that if φ is non-wasteful, there is one of such continuations for which at least k violations can be created.

Therefore, Equations (1) to (4) imply that

$$E(P^*_{2.2}) + F(P^*_{2.2}) \geq 2 + k > k + 1. \qquad \square$$

We prove Theorem 2 using Lemma A.1. Given $P \in \mathcal{P}$, let $n \equiv |I|$. If $n = 0$, there are no applicants, so trivially $V(\varphi, 0) = 0 = \lfloor \frac{0}{2} \rfloor$. If $n = 1$, there is only one applicant and no ex-post Pareto efficiency or elimination of justified envy violations exist, so $V(\varphi, 1) = 0 = \lfloor \frac{1}{2} \rfloor$. In general, note that $V(\varphi, n) \geq E(\varphi(P)) + F(\varphi(P))$ for all $P \in \mathcal{P}$ such that $|I| = n$. This is particularly true when $n = 2k$ for some $k \in \mathbb{N}$, any P in an arbitrary order-k PHAP test set. Then, Lemma A.1 implies that $V(\varphi, n) \geq \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$ when n is even.

It is left to establish the result for n odd, which follows a construct analogous to the proof of lemma A.1. Suppose that $n = 2k + 1$ for some $k \in \mathbb{N}$. Consider a new type of PHAP test set $\hat{\mathcal{P}}(k)$ which has the characteristics of an order-k PHAP test set, with the only difference that it features $k + 1$ λ-type applicants and k ω-type applicants, with preferences $(\Omega, 2k+1) \succ_\omega (\Lambda, 0)$ for all ω and $(\Lambda, 2k+1) \succ_\lambda (\Omega, 0)$ for all λ.

Analogously to the proof of Lemma A.1, when $a_0 = \Omega$, there are two cases. First, if there is λ_0 such that $i_{\varphi(P)}(a_0) = \lambda_0$ for all $P \in \hat{\mathcal{P}}(k)$, the proof is similar to Case 2.2 in Lemma A.1. The continuation of the problem from $t = 1$ onward forms an order-k PHAP test set as in Definition A.1, and we can construct a valid mechanism to solve the continuation problem. Then, Lemma A.1 implies that there exists one continuation problem that creates at least k violations to ex-post Pareto efficiency and elimination of justified envy, so $V(\varphi, 2k+1) \geq \lfloor \frac{2k+1}{2} \rfloor = k$. Second, if there is ω_0 such that $i^\varphi_P(a_0) = \omega_0$ for all $P \in \hat{\mathcal{P}}(k)$, an argument similar to Case 1 in the proof of Lemma A.1 ensues. Specifically, in a PHAP $P \in \hat{\mathcal{P}}$ featuring $a_t = \Omega$ for all $t \in \{1, 2, \ldots, 2k\}$ implies that all the $k+1$ λ-type applicants receive a unit in Ω at a time later than $t = 0$. As $\lambda_i \rhd^*_\Omega \omega_0$, there are $k + 1$ applicants that justifiably envy ω_0. We conclude that $V(\varphi, 2k+1) \geq k + 1 > k = \lfloor \frac{2k+1}{2} \rfloor$. In either case, if n is odd, $V(\varphi, n) \geq \lfloor \frac{n}{2} \rfloor$. $\square$

Proof of Claim A.1. Consider the setup of Case 2.2 in the proof of Lemma A.1. For a given

$P \in \mathcal{P}_1$, define $\tilde{P}$ to be a function that removes ω_1, λ_0, a_0 and a_1 from P:

$$\tilde{P}(P) = (I(k+1) \setminus I_S, B^*, \succ^*, \triangleright^*, \pi^*, (a_t)_{t\geq 2}).$$

Let $\tilde{\mathcal{P}}_1 = \left\{\tilde{P}(P) : P \in \mathcal{P}_1\right\}$, the set of continuation problems generated by $\mathcal{P}_1$. To apply Equation (IH), we first must verify that $\tilde{\mathcal{P}}_1$ is an order-k PHAP test set.

Step 1. $\tilde{\mathcal{P}}_1$ is an order-k PHAP test set.

Note that if $P \in \mathcal{P}_1$, then $I = I(k+1)$, $B = B^*$, $\succ = \succ^*$, $\triangleright = \triangleright^*$, $\pi = \pi^*$, the elements of the test PHAP set $\mathcal{P}(k+1)$ we are focusing on in the inductive step of lemma A.1. Moreover, if $P_1 = \mathcal{P}_1$, $a \in A(\pi^*)$ with $a_0 = \Omega$ and $a_1 = \Lambda$. Therefore, we can rewrite

$$\tilde{\mathcal{P}}_1 = \left\{(I, B, \succ, \triangleright, \pi, a) : I = I(k+1) \setminus \{\omega_1, \lambda_0\}, B = B^*, \succ = \succ^*, \triangleright = \triangleright^*, \pi = \pi^*, a \in \tilde{A}\right\},$$

with $\tilde{A} = \{\tilde{a} : \tilde{a} = (a_t)_{t\geq 2}$ for some $a \in A(\pi^*)$ s.t. $a_0 = \Omega, a_1 = \Lambda\}$. We verify that the elements of $\mathcal{P}_1$ are the elements of a PHAP test set per definition A.1:

- **Applicants:** Because $\mathcal{P}(k+1)$ is an order-$k+1$ PHAP test set, $I(k+1) = I_\lambda(k+1) \cup I_\omega(k+1)$, where $I_\omega(k+1)$ is a set of $k+1$ ω-type applicants and $I_\lambda(k+1)$ is a set of $k+1$ λ-type applicants. Then $I_\omega(k) = I_\omega(k+1) \setminus \{\omega_1\}$ and $I_\lambda(k) = I_\lambda(k+1) \setminus \{\lambda_0\}$ are sets of k ω-type and λ-type applicants. By writing $I(k) \setminus \{\lambda_0, \omega_1\} = I_\omega(k) \cup I_\lambda(k)$, we see that $I(k)$ has the desired properties.
- **Buildings:** B^* is an arbitrary set of 2 buildings, namely, $B^* = \{\Omega, \Lambda\}$.
- **Preferences:** Costly waiting implies that $(\Omega, 2k) \succ^*_\omega (\Omega, 2(k+1)) \succ^*_\omega (\Lambda, 0)$ for all $\omega \in I_\omega(k)$. Likewise, $(\Lambda, 2k) \succ^*_\lambda (\Lambda, 2(k+1)) \succ^*_\lambda (\Omega, 0)$ for all $\lambda \in I_\lambda(k)$.
- **Priorities:** The original priorities $\triangleright^*$ are such that $\lambda \triangleright^*_\Omega \omega$ and $\omega \triangleright^*_\Lambda \lambda$ for all $\omega \in I_\omega(k)$ and $\lambda \in I_\lambda(k)$.
- **Distribution of Arrivals:** As π^* is a MAP, a_t and a'_t are independent and identically distributed for $t, t' \in \{0, 1, \ldots\}$. Therefore, π^* is still a MAP distribution over arrival tails $(a_t)_{t\geq 2}$.
- **Realized Arrivals:** To show that $\tilde{\mathcal{P}}$ is a valid test set, we must show that $\tilde{A} = A(\pi^*)$, which implies also that $\tilde{\mathcal{P}} \subset \mathcal{P}$.

  To see that $\tilde{A} \subset A(\pi^*)$, let $\tilde{a} \in \tilde{A}$. Then, there exists $a \in A(\pi^*)$ with $a_0 = \Omega$ and $a_1 = \Lambda$ such that $\tilde{a} = (a_{t'})_{t'\geq 2}$. Note that $a_{t'} \in \{\Omega, \Lambda\}$ if $t' \in \{2, 3, \ldots\}$ and $a_{t'} = \emptyset$ otherwise. By defining $t = t' - 2$, we can write $\tilde{a} = (a_t)_{t\geq 0}$ with $a_t \in \{\Omega, \Lambda\}$ if $t \in \mathbb{N}$ and $a_t = \emptyset$ otherwise, which implies that $a \in A(\pi^*)$.

  To see that $\tilde{A} \supset A(\pi^*)$, let $a \in A(\pi^*)$. This means that $a = (a_t)_{t\geq 0}$, with $a_t \in \{\Omega, \Lambda\}$ if $t \in \mathbb{N}_0$ and $a_t = \emptyset$ otherwise. Define a sequence a' as follows: $a'_t = \emptyset$ if $t \notin \mathbb{N}_0$, $a'_0 = \Omega$, $a'_1 = \Lambda$, and $a'_t = a_{t-2}$ for all $t \in \{2, 3, \ldots\}$. Clearly, $a'_t \in \{\Lambda, \Omega\}$ when $t \in \mathbb{N}_0$ and

$a'_t = \emptyset$ otherwise, so $a' \in A(\pi^*)$. Moreover, $a = (a')_{t\geq 2}$, so $a \in \tilde{A}$.

Q.E.D. Step 1

One implication of Step 1 is that $\tilde{\mathcal{P}}_1 \subset \mathcal{P}$, so any allocation mechanism can solve a continuation problem $P \in \tilde{\mathcal{P}}_1$ as a stand-alone valid PHAP.

Step 2. Suitably define $\tilde{\varphi}$ such that $E(\tilde{\varphi}(\tilde{P}^*)) + F(\tilde{\varphi}(\tilde{P}^*)) \geq k$ for some $\tilde{P}^* \in \tilde{\mathcal{P}}_1$. Consider φ, the non-wasteful, time-consistent, and individually rational mechanism of interest in Case 2.2 of Lemma A.1. Define mechanism $\tilde{\varphi}$ as the mechanism that allocates units in the continuation of $P \in \mathcal{P}_1$ in the same way φ does in the full problem. Formally, whenever $\tilde{P}(P) \in \tilde{\mathcal{P}}_1$, $\tilde{\varphi}_{\tilde{P}(P)}(i) = \varphi_P(i)$ for all $i \in I(k+1) \setminus I_S$; meanwhile, $\tilde{\varphi}(P)$ is any non-wasteful, time-consistent, and individually rational matching outcome when $P \in \mathcal{P} \setminus \tilde{\mathcal{P}}_1$.

Per Step 1, as $\tilde{\mathcal{P}}_1 \subset \mathcal{P}_1$, $\tilde{\varphi}$ is well defined as a matching mechanism. Moreover, as φ is a non-wasteful mechanism, $\tilde{\varphi}$ is as well. Therefore, from Step 1 and equation (IH) it follows that

$$\exists \tilde{P}^* \in \tilde{\mathcal{P}}_1 \text{ s.t. } E(\tilde{\varphi}(\tilde{P}^*)) + F(\tilde{\varphi}(\tilde{P}^*)) \geq k. \tag{5}$$

Q.E.D. Step 2

Step 3. Use $\tilde{\varphi}$ to show that there is $P^*_{2.2} \in \mathcal{P}_1$ s.t. $E(\varphi(P^*_{2.2})) + F(\varphi(P^*_{2.2})) \geq k+1$. As $\tilde{P}^* \in \tilde{\mathcal{P}}_1$, there must exist $P^*_{2.2} \in \mathcal{P}_1$ such that $\tilde{P}^*$ is the continuation of $P^*_{2.2}$, namely, $\tilde{P}^* = \tilde{P}(P^*_{2.2})$. By construction, every $i \in I(k+1) \setminus I_S$ features in $\tilde{P}^*$ and $P^*_{2.2}$ the same preferences and $\tilde{\varphi}_{\tilde{P}^*}(i) = \varphi_{P^*_{2.2}}(i)$.

$$F(\tilde{\varphi}(\tilde{P}^*)) = F(\varphi(P^*_{2.2}) \mid I(k+1) \setminus I_S). \tag{6}$$

We define $\psi(\tilde{P}^*)$ as the matching outcome with most Pareto efficiency violations:

$$\psi(\tilde{P}^*) = \underset{\phi(\tilde{P}^*) \in \mathcal{D}(\tilde{\varphi}(\tilde{P}^*)) \cup \left\{\tilde{\varphi}(\tilde{P}^*)\right\}}{\arg\max} |\{i \in I : \phi_{\tilde{P}^*}(i) \succ_i \varphi_{\tilde{P}^*}(i)\}|.$$

Then, we define $\phi^*(P^*_{2.2})$ to be the matching outcome that swaps the matching outcomes of ω_1 and λ_0 under $\varphi(P^*_{2.2})$ while assigning the rest of the units as ψ, namely, $\phi^*_{P^*_{2.2}}(\omega_1) = \varphi_{P^*_{2.2}}(\lambda_0)$, $\phi^*_{P^*_{2.2}}(\lambda_0) = \varphi_{P^*_{2.2}}(\omega_1)$, and $\phi^*_{P^*_{2.2}}(i) = \psi_{\tilde{P}^*}(i)$ for all $i \in I(k+1) \setminus I_S$. By construction,

$$E(\tilde{\varphi}(\tilde{P}^*)) = E(\varphi(P^*_{2.2}), \phi^*(P^*_{2.2}) \mid I(k+1) \setminus I_S). \tag{7}$$

As $\tilde{P}^* = \tilde{P}(P^*_{2.2})$, Equations (5) to (7), imply that

$$E(\varphi(P^*_{2.2}), \phi^*(P^*_{2.2}) \mid I(k+1) \setminus I_S) + F(\varphi(P^*_{2.2}) \mid I(k+1) \setminus I_S) \geq k. \tag{8}$$

Finally, by construction, $\phi^*(P^*_{2.2}) \in \mathcal{D}^*(\varphi(P^*_{2.2}))$, so, given that $P^*_{2.2} \in \mathcal{P}_1$, Equation (3) applies. Then, Equations (1) to (4) imply that

$$E(P^*_{2.2}) + F(P^*_{2.2}) \geq 2 + k > k + 1.$$

Q.E.D. Step 3 □

A.3 Proof of Theorem 3

Proof of Theorem 3. Non-wastefulness follows directly by construction, as the rules of the CBWL algorithm imply that no unit is discarded when an individually rational matching remains in L, and every unit is delivered upon arrival. To show the rest of the properties, we must prove the following auxiliary result that outlines the consequences of acyclicity:

Lemma A.2. *Consider $(I, B, \rhd, \succ, \pi, A(\pi)) \in \mathcal{P}$. Assume that $\rhd$ is acyclic and there exists an applicant $i \in L \subset I$, acceptable for $\beta \in B$, with priority $i \rhd_{\beta^*} i'$ for all $i' \in L \setminus \{i\}$. If i is acceptable for $\beta' \in B \setminus \{\beta\}$, then, at most one applicant $i' \in L \setminus i$, acceptable for β, has a priority $i' \rhd_{\beta'} i$. Moreover, if i is acceptable for $\beta'' \in B \setminus \{\beta, \beta'\}$ and there is an applicant $i'' \in L \setminus i$, acceptable for β, with a priority $i'' \rhd_{\beta''} i$, then $i'' = i'$.*

Proof of Lemma A.2. Assume that $\rhd$ is acyclic and there exists a building $\beta \in B$ with an acceptable applicant $i \in L \subset I$ such that $i \rhd_{\beta^*} i'$ for all $i' \in L \setminus \{i\}$. Furthermore, assume that $i \rhd_{\beta'} \beta'$ for some $\beta' \in B \setminus \{\beta\}$, but that there are $n \geq 2$ applicants $i_1, \ldots, i_n \in L$ such that $i_k \rhd_\beta \beta$ and $i_k \rhd_{\beta'} i$ for all $k = 1, \ldots, n$. Among such n applicants, we can find i_{k_1} and i_{k_2} such that $i_{k_1} \rhd_{\beta'} i_{k_2} \rhd_{\beta'} i \rhd_{\beta'} \beta'$. However, as $i \rhd_\beta i_{k_1} \rhd_\beta \beta$ by assumption, we have a contradiction with $\rhd$ being acyclic. We conclude that at most one applicant $i' \in L$ acceptable at β has a priority $i' \rhd_{\beta'} i$.

For the second part of the result, suppose that $i \rhd_{\beta''} \beta''$ for some $\beta'' \in B \setminus \{\beta, \beta'\}$ and there is an applicant $i'' \in L \setminus i$ such that $i'' \rhd_\beta \beta$ and $i'' \rhd_{\beta''} i$. Moreover, assume that $i'' \neq i'$. Without loss of generality, assume that $i' \rhd_\beta i''$, as the opposite case is analogous. It follows, from the definition of i, i', and i'', that $i \rhd_\beta i' \rhd_\beta i'' \rhd_\beta \beta$ and $i'' \rhd_{\beta''} i \rhd_{\beta''} \beta''$, a contradiction with $\rhd$ being acyclic. We conclude that $i' = i''$ is the only possibility. □

Strategy-proofness follows directly from Lemma A.2:

Strategy-proofness. No applicant can affect their own assignment time by misreporting their preferences, as the mechanisms pick applicants based on their priorities and other applicants' reported preferences. Moreover, the last applicant in L cannot strictly improve by misreporting their preferences, as the only potential consequence of doing so is receiving a unit that is not preferred over remaining unmatched. It follows that, when $|L| \geq 2$, an applicant i can only affect their outcome through their reported preference $\tilde{\succ}_i$ in two cases: when $i^* = i$ and when $i' = i$. Per the CBWL algorithm, when $i' = i$, the applicant gets their most desirable queue with respect to the reported preferences. Hence, no misreport can ever strictly improve their outcome.

Consider the case where $i^* = i$. We denote by $\beta^*(\tilde{\succ}_i)$ the applicant's top building according to their reported preferences and $i'(\tilde{\succ}_i) \in L$ the subsequent top-priority applicant at that building. Let $B_i \subset B$ be the buildings i is acceptable for. Following from lemma A.2, as i is $\beta^*(\tilde{\succ}_i)$'s top applicant, we can subdivide B_i into the set B_i^* where i has the top priority, and the set $B_i \setminus B_i^*$, where they have the second-highest priority, only topped by $i'(\tilde{\succ}_i)$. Let $\succ_i$ be applicant i's true preference.

If $\beta^*(\succ_i) \in B_i^*$, the applicant cannot strictly improve by misreporting their preferences because being a top-priority applicant implies that the CBWL algorithm places them in their desired queue immediately.

Finally, when $\beta^*(\succ_i) \in B_i \setminus B_i^*$, i may want to misreport their preferences for two reasons. The first one is to influence the choice of $i'(\tilde{\succ}_i)$. However, lemma A.2 implies that $i'(\tilde{\succ}_i)$ is the same applicant for all $\beta \in B_i \setminus B_i^*$. Therefore, i has no benefit from misreporting their preferences to influence $i'(\tilde{\succ}_i)$.

The second reason to misreport is to secure a queue other than $\beta^*(\succ_i)$. In particular, $Q_{\beta^*(\succ_i)}$ is i's preferred queue in the current state, this is, without $i'(\succ_i)$ being added to any queue. However, it is possible that i's top choice is $Q_{\beta'}$ after $i'(\succ_i)$ has been added to a queue.[1] Anticipating that the addition of $i'(\succ_i)$ to a queue may affect their preferences, applicant i could attempt to immediately secure $Q_{\beta'}$ by misreporting $\succ_i$.[2] We argue that a strict improvement in this way is not attainable.

Suppose that by misreporting their preferences, applicant i can secure queue $Q_{\beta'}$ in the current state, which tops any other queue, anticipating that $i'(\succ_i)$ will be assigned to a queue before i. On the other hand, if i truthfully reports their preferences, $i'(\succ_i)$ is added to the end of their desired queue. Then, applicant $i^* = i$ is selected again immediately after, due to lemma A.2, which implies that i now has the top priority at every building among

[1] For instance, if $i'(\succ_i)$ wants $Q_{\beta^*(\succ_i)}$, the expected waiting time after adding an extra applicant may sway the preferences of i to a different queue.

[2] This is possible, for instance, when $\beta' \in B_i^*$.

$L \setminus \{i'(\succ_i)\}$. As i strictly prefers $Q_{\beta'}$ over any other queue after $i'(\succ_i)$ has been assigned to one, the mechanism places i at the end of $Q_{\beta'}$. Hence, i is indifferent between misreporting their preferences to secure $Q_{\beta'}$ before $i'(\succ_i)$ is assigned to a queue and reporting their true preferences to secure $Q_{\beta'}$ after $i'(\succ_i)$ is assigned to a queue.

Given mechanism φ, $P \in \mathcal{P}$, and $t' \in \mathcal{T}$, let $M(\varphi(P), t')$ be the set of units assigned under $\varphi(P)$ prior to period t':

$$M(\varphi(P), t') = \{\langle \beta_{\varphi(P)}(i), r_{\varphi(P)}(i)\rangle : i \in I \text{ s.t. } T_{\varphi(P)}(i) < t'\}$$

To show the elimination of justified envy and Pareto efficiency, we rely on the following lemma:

Lemma A.3. *Suppose that there exists $T^* \in \mathcal{T}$, $P \in \mathcal{P}$, and a mechanism ϕ such that $M(\varphi^{\text{CBWL}}(P), T^*) = M(\phi(P), T^*)$. The following two statements hold:*

i. If there are $i_1, i_2 \in I$ such that $T_{\varphi^{\text{CBWL}}(P)}(i_2) = T^* < T_{\phi(P)}(i_1)$ and $i_2 \rhd_{\beta_{\phi(P)}(i_1)} \beta_{\phi(P)}(i_1)$, then $\overline{\varphi}^M_P(i_2, T^*) \succ_{i_2} \overline{\phi}_P(i_1, T^*)$.

ii. $\langle \beta, r\rangle$ is an unassigned unit in $\varphi^{\text{CBWL}}(P)$ and there is $i_2 \in I$ such that $T_{\varphi^{\text{CBWL}}(P)}(i_2) = T^*$ and $i_2 \rhd_\beta \beta$, then $\overline{\varphi}^M_P(i_2, T^*) \succ_{i_2} (\langle \beta, r\rangle, T^* + \tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta, r\rangle, T^*)$.

Proof of lemma A.3. Part (i): If $i_2 \rhd_{\beta_{\phi(P)}(i_1)} \beta_{\phi(P)}(i_1)$, φ^{CBWL} considers $Q_{\beta_{\phi(P)}(i_1)}$ for i_2, if selected for assignment. Placing i_2 in $Q_{\beta_{\phi(P)}(i_1)}$ at time T^* amounts to assigning i_2 a unit $\langle \beta_{\varphi^{\text{CBWL}}(P)}(i_1), r_{i_2}\rangle$ for some $r_{i_2} \in \mathbb{N}$. Due to strategy-proofness, when the CBWL algorithm places an applicant on a queue, such a queue is preference-maximizing among queues where the applicant is acceptable. Therefore, we must have

$$\overline{\varphi}^M_P(i_2) \succeq_{i_2} \Big(\langle \beta_{\varphi^{\text{CBWL}}(P)}(i_1), r_{i_2}\rangle, T_{\varphi^{\text{CBWL}}(P)}(i_2) + \tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta_{\varphi^{\text{CBWL}}(P)}(i_1), r_{i_2}\rangle, T_{\varphi^{\text{CBWL}}(P)}(i_2))\Big). \tag{9}$$

Now, as $M(\varphi^{\text{CBWL}}(P), T^*) = M(\phi(P), T^*)$, both mechanisms have assigned the same units up to time T^*. Therefore, as $T^* = T_{\varphi^{\text{CBWL}}(P)}(i_2) = T^* < T_{\phi(P)}(i_1)$, $r_{i_2} \leq r_{\phi(P)}(i_2)$, as other applicants may be awarded additional units at $\beta_{\phi(P)}(i)$ between periods. Consequently, as φ^{CBWL} is a non-wasteful mechanism and no mechanism can deliver a unit that has not yet arrived, we must have that

$$\tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta_{\varphi^{\text{CBWL}}(P)}(i_1), r_{i_2}\rangle, T_{\varphi^{\text{CBWL}}(P)}(i_2)) \leq \tau_{\phi(P)}(\mu_{\phi(P)}(i_1), T_{\varphi^{\text{CBWL}}(P)}(i_2)). \tag{10}$$

If $\beta_{\varphi^{\text{CBWL}}(P)}(i_2) = \beta_{\phi(P)}(i_1)$, then $r_{i_2} < r_{\phi(P)}(i_1)$ and Equation (10) holds strictly. Meanwhile, if $\beta_{\varphi^{\text{CBWL}}(P)}(i_2) \neq \beta_{\phi(P)}(i_1)$, applicants' strict preferences imply that Equation (9) holds strictly. Either way, then, Equation (9), Equation (10), preferences' costly-waiting property, and $T^* = T_{\varphi^{\text{CBWL}}(P)}(i_2)$ imply that $\overline{\varphi}^M_P(i_2, T^*) \succ_{i_2} \overline{\phi}_P(i_1, T^*)$.

Part (ii): The proof is analogous to Part (i). At T^*, the CBWL algorithm considers Q_β for i_2, so there exists $r_{i_2} \in \mathbb{N}$ such that

$$\overline{\varphi}^M_P(i_2) \succeq_{i_2} \Big(\langle \beta, r_{i_2} \rangle, T_{\varphi^{\text{CBWL}}(P)}(i_2) + \tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta, r_{i_2} \rangle, T_{\varphi^{\text{CBWL}}(P)}(i_2))\Big). \tag{11}$$

Moreover, as applicants may be assigned to Q_β after T^*,

$$\tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta, r_{i_2} \rangle, T^*) \leq \tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta, r \rangle, T^*). \tag{12}$$

If $\beta_{\varphi^{\text{CBWL}}(P)}(i_2) = \beta$, then Equation (12) holds strictly. If $\beta_{\varphi^{\text{CBWL}}(P)}(i_2) \neq \beta$, equation (11) holds strictly. Either way, then, Equation (11), Equation (12), preferences' costly-waiting property, and $T^* = T_{\varphi^{\text{CBWL}}(P)}(i_2)$ imply that $\overline{\varphi}^M_P(i_2, T^*) \succ_{i_2} (\langle \beta, r \rangle, T^* + \tau_{\varphi^{\text{CBWL}}(P)}(\langle \beta, r \rangle, T^*))$.

□

Elimination of justified envy. If we choose $\phi = \varphi^{\text{CBWL}}$, we trivially have that for all $t \in \mathcal{T}$, $M(\varphi^{\text{CBWL}}, t) = M(\phi, t)$. Therefore, Part (i) in Lemma A.3 implies that for any $i_1, i_2 \in I$ such that $T_{\varphi^{\text{CBWL}}(P)}(i_2) < T_{\varphi^{\text{CBWL}}(P)}(i_1)$ and $i_2 \triangleright_{\beta_{\varphi^{\text{CBWL}}(P)}(i_1)} \beta_{\varphi^{\text{CBWL}}(P)}(i_1)$, then $\overline{\varphi}^M_P(i_2, T_{\varphi^{\text{CBWL}}(P)}(i_2)) \succ_{i_2} \overline{\varphi}^M_P(i_1, T_{\varphi^{\text{CBWL}}(P)}(i_2))$. As a consequence, i_2 cannot justifiably envy i_1 when $T_{\varphi^{\text{CBWL}}(P)}(i_2) < T_{\varphi^{\text{CBWL}}(P)}(i_1)$.

Suppose, then, that for some $i_1, i_2 \in I$, $T_{\varphi^{\text{CBWL}}(P)}(i_2) \geq T_{\varphi^{\text{CBWL}}(P)}(i_1)$. Two cases arise. In the first case, $T_{\varphi^{\text{CBWL}}(P)}(i_2) > T_{\varphi^{\text{CBWL}}(P)}(i_1)$, where it follows that $i_1, i_2 \in L$ at the beginning of time $T_{\varphi^{\text{CBWL}}(P)}(i_1)$. Consider the two ways i_1 can get an assignment: by being i^* in Step 3 or by being i' in Step 4. If $i^* = i_1$, Lemma A.2 implies that at most only other applicant $i' \in L$ can have a higher priority at a given building, potentially leading to justified envy. If $i' = i_2$, Step 4 of the CBWL algorithm implies that i_2 is placed at this point on their most desirable queue where acceptable, leading to $T_{\varphi^{\text{CBWL}}(P)}(i_1) = T_{\varphi^{\text{CBWL}}(P)}(i_2)$. On the other hand, if $i' = i_1$, through Lemma A.2 implies that there is only another applicant $i^* \in L$ with a higher priority at a given building. A possibility is that $i^* = i_2$, case in which the fact that $T_{\varphi^{\text{CBWL}}(P)}(i_1) < T_{\varphi^{\text{CBWL}}(P)}(i_2)$ implies, via Steps 3 and 4 of the CBWL algorithm, that i_2 preferred over Q_{β_A} a queue for which i_1 had a higher priority, and that i_1 is acceptable and desires the most queue Q_{β_A} instead. However, applicants' strict preferences imply in turn that $\overline{\varphi}^M_P(i_2, T_{\varphi^{\text{CBWL}}(P)}(i_2)) \succ_{i_2} \overline{\varphi}^M_P(i_1, T_{\varphi^{\text{CBWL}}(P)}(i_2))$, ruling out the possibility of interim justified envy from i_2 to i_1.

In the second case, $T_{\varphi^{\text{CBWL}}(P)}(i_2) = T_{\varphi^{\text{CBWL}}(P)}(i_1)$. There are only two possibilities stemming from Steps 3 and 4 of the CBWL algorithm: $i^* = i_1$ and $i' = i_2$, or $i^* = i_2$ and $i' = i_1$. On the one hand, if $i^* = i_1$ and $i' = i_2$, i_2 cannot justifiably envy i_1 because Step 4 in the CBWL algorithm implies that i_2 is placed at their most desirable queue where they are acceptable, implying that if acceptable at $\beta_{\varphi^{\text{CBWL}}(P)}(i_1)$, then $\overline{\varphi}_P^M(i_2, T_{\varphi^{\text{CBWL}}(P)}(i_2)) \succ_{i_2} \overline{\varphi}_P^M(i_1, T_{\varphi^{\text{CBWL}}(P)}(i_2))$. On the other hand, $i^* = i_2$ and $i' = i_1$ implies either, via Step 4, that i_1 has the highest priority at their building, or via Step 3, that i_2 is not acceptable at $\beta_{\varphi^{\text{CBWL}}(P)}(i_1)$, or it is acceptable but prefers a different queue, hence, $\overline{\varphi}_P^M(i_2, T_{\varphi^{\text{CBWL}}(P)}(i_2)) \succ_{i_2} \overline{\varphi}_P^M(i_1, T_{\varphi^{\text{CBWL}}(P)}(i_2))$.

Pareto Efficiency. Consider any mechanism ϕ and the set of applicants for which ϕ's expected assignment differs from the expected assignment under φ^{CBWL}:

$$D = \{i \in I : \overline{\varphi}_P^M(i, T_{\varphi^{\text{CBWL}}(P)}(i)) \neq \overline{\phi}_P(i, T_{\phi^M(P)}(i))\}.$$

We define $T^* = \min_{i \in D} T_{\varphi^{\text{CBWL}}(P)}(i)$ and $I^* = \arg\min_{i \in D} T_{\varphi^{\text{CBWL}}(P)}(i)$. By construction, $M(\varphi^{\text{CBWL}}(P), T^*) = M(\phi(P), T^*)$. Consider any applicant in $i_2 \in I^*$. If $\mu_{\phi(P)}(i_2)$ is an unassigned unit under $\varphi^{\text{CBWL}}(P)$, Part (ii) of Lemma A.3 implies that $\overline{\varphi}_P^M(i_2) \succ_{i_2} \overline{\phi}_P(i_2)$, as the individual rationality of the latter matching outcome implies that $i_2 \rhd_{\beta_{\phi(P)}(i_2)} \beta_{\phi(P)}(i_2)$. Therefore φ^{CBWL} must be interim Pareto efficient with respect to ϕ, as i_2 strictly prefers their expected outcome under $\varphi^{\text{CBWL}}(P)$.

If $\mu_{\phi(P)}(i_2)$ is assigned under $\varphi^{\text{CBWL}}(P)$, the fact that $i_2 \in I^*$ implies that there exists $i_1 \in I$ such that $\mu_{\varphi^{\text{CBWL}}(P)}(i_1) = \mu_{\phi(P)}(i_2)$. Note that $T_{\varphi^{\text{CBWL}}(P)}(i_1) < T^*$ is not possible because $M(\varphi^{\text{CBWL}}(P), T^*) = M(\phi(P), T^*)$. Two cases remain, then. First, if $T_{\varphi^{\text{CBWL}}(P)}(i_1) > T^*$,

$$\overline{\varphi}_P^M(i_2) \succ_{i_2} (\mu_{\varphi^{\text{CBWL}}(P)}(i_1), T^* + \tau_{\varphi^{\text{CBWL}}(P)}(\mu_{\varphi^{\text{CBWL}}(P)}(i_1), T^*)) \succeq_{i_2} \overline{\phi}_P(i_2),$$

where the first inequality follows from Part (i) of Lemma A.3 comparing the CBWL algorithm to itself, and the second inequality follows from preferences' costly-waiting property and that the CBWL algorithm is non-wasteful, so given that $M(\varphi^{\text{CBWL}}(P), T^*) = M(\phi(P), T^*)$, we have $\tau_{\varphi^{\text{CBWL}}(P)}(\mu_{\varphi^{\text{CBWL}}(P)}(i_1), T^*)) \leq \tau_{\varphi^{\text{CBWL}}(P)}(\mu_{\phi^M(P)}(i_1), T^*))$. Therefore φ^{CBWL} must be interim Pareto efficient with respect to ϕ, as i_2 strictly prefers their expected outcome under $\varphi^{\text{CBWL}}(P)$.

Second, the case of $T_{\varphi^{\text{CBWL}}(P)}(i_1) = T^*$. Due to the rules of the CBWL algorithm, it must be that i_2 plays the role of i^* and i_1 plays the role of i' or vice versa. The argument is symmetric, so assume $i^* = i_2$. Due to ϕ's individual rationality, it must be that $i_2 \rhd_{\beta_{\varphi^{\text{CBWL}}(P)}(i_1)} \beta_{\varphi^{\text{CBWL}}(P)}(i_1)$, and so the CBWL algorithm considers $Q_{\beta_{\varphi^{\text{CBWL}}(P)}(i_1)}$ for i_2. Instead, the mechanism places i_2 in $Q_{\beta_{\varphi^{\text{CBWL}}(P)}(i_2)}$. Then,

$$\overline{\varphi}_P^M(i_2) \succ_{i_2} (\mu_{\varphi^{\text{CBWL}}(P)}(i_1), T^* + \tau_{\varphi^{\text{CBWL}}(P)}(\mu_{\varphi^{\text{CBWL}}(P)}(i_1), T^*)) \succeq_{i_2} \overline{\phi}_P(i_2),$$

Where the first inequality follows from $i^* = i_2$, Step 4 of the mechanism and applicants' strict preferences, and the second inequality follows from the preferences' costly-waiting property and CBWL mechanism's non-wastefulness, so given that $M(\varphi^{\text{CBWL}}(P), T^*) = M(\phi(P), T^*)$, we have $\tau_{\varphi^{\text{CBWL}}(P)}(\mu_{\varphi^{\text{CBWL}}(P)}(i_1), T^*)) \leq \tau_{\varphi^{\text{CBWL}}(P)}(\mu_{\phi^M(P)}(i_1), T^*))$. We conclude that i_2 strictly prefers their CBWL expected outcome under $\varphi^{\text{CBWL}}(P)$. □

A.4 Proof of Theorem 4

Proof of Theorem 4. Suppose that $\triangle$ is not an acyclic priority space, so there exists a priority profile $\rhd^* \in \triangle$, applicants $i_1, i_2, i_3 \in \mathcal{I}$, and buildings $\beta_1, \beta_2 \in \mathcal{B}$ such that $i_1 \rhd^*_{\beta_1} i_2 \rhd^*_{\beta_1} i_3 \rhd^*_{\beta_1} \beta_1$ and $i_3 \rhd^*_{\beta_2} i_1 \rhd^*_{\beta_2} \beta_2$. Based on these applicants and buildings we define the sets $I^* = \{i_1, i_2, i_3\}$ and $B^* = \{\beta_1, \beta_2\}$.

Consider the following associated PHAP P^*. Buildings and applicants are B^* and I^*; arrivals happen with certainty, but only during periods $t = 0, 1, 2$, with $a_0 = \beta_1$, $a_1 = \beta_2$, and $a_2 = \beta_1$; buildings' priorities are $\rhd^*$; finally, both buildings are acceptable for all applicants, who have the following preferences:

$$i_1 : (\beta_2, 0) \succ (\beta_2, 1) \succ (\beta_1, 0) \succ (\beta_2, 2) \succ (\beta_1, 1) \succ (\beta_1, 2)$$
$$i_2 : (\beta_1, 0) \succ (\beta_1, 1) \succ (\beta_2, 0) \succ (\beta_1, 2) \succ (\beta_2, 1) \succ (\beta_2, 2)$$
$$i_3 : (\beta_2, 0) \succ (\beta_1, 0) \succ (\beta_2, 1) \succ (\beta_1, 1) \succ (\beta_2, 2) \succ (\beta_1, 2)$$

Consider any non-wasteful mechanism φ so units cannot be discarded or withheld beyond their arrival period, given that every potential matching is individually rational. φ's non-wastefulness and the deterministic arrival process, which makes the arrival and delivery time known, implies that $t_{\varphi(P^*)}(i) = T_{\varphi(P^*)}(i) + \tau_{\varphi(P^*)}(\mu_{\varphi(P^*)}(i), T_{\varphi(P^*)}(i))$. Consequently, $\overline{\varphi}_{P^*}(i, T_{\varphi(P^*)}(i)) = \varphi_{P^*}(i)$ for all $i \in I$. With this observation, we show that an interim Pareto improvement or case of justified envy exists regardless of the matching outcome.

In a first case, $i_{\varphi(P^*)}(a_0) = i_1$, $i_{\varphi(P^*)}(a_1) = i_2$, and $i_{\varphi(P^*)}(a_2) = i_3$. Then, $\varphi(P^*)$ is ex-ante Pareto dominated by the matching outcome $\phi(P^*)$ that swaps the assignments of i_1 and i_2, leaving i_3 unaffected, delivering the units as they become available. As $\phi(P^*)$ is non-wasteful, we have that $\overline{\phi}_{P^*}(i, T_{\phi(P^*)}(i)) = \phi_{P^*}(i)$ for all $i = i_1, i_2, i_3$. Then, $\overline{\phi}_{P^*}(i_1, T_{\phi(P^*)}(i_1)) = (\langle\beta_2, 1\rangle, 1) \succ^*_{i_1} (\langle\beta_1, 1\rangle, 0) = \overline{\varphi}_{P^*}(i_1, T_{\varphi(P^*)}(i_1))$ and $\overline{\phi}_{P^*}(i_2, T_{\phi(P^*)}(i_2)) = (\langle\beta_1, 1\rangle, 0) \succ^*_{i_2} (\langle\beta_2, 1\rangle, 1) = \overline{\varphi}_{P^*}(i_2, T_{\varphi(P^*)}(i_2))$.

It can be verified that analogously, every other possible case leads to the following violations:

	$a_0 = \beta_1$	$a_1 = \beta_2$	$a_2 = \beta_1$	Pareto	Justified Envy
Case 1	i_1	i_2	i_3	$i_1 \leftrightarrow i_2$	
Case 2	i_1	i_3	i_2	$i_1 \leftrightarrow i_3$	
Case 3	i_2	i_1	i_3		$i_3 \to i_1$
Case 4	i_2	i_3	i_1		$i_1 \to i_2$
Case 5	i_3	i_1	i_2		$i_2 \to i_3$
Case 6	i_3	i_2	i_1		$i_1 \to i_3$, $i_2 \to i_3$

□

A.5 Proof of Theorem 5

Proof of Theorem 5. Suppose φ is a non-wasteful mechanism. We use the counter-examples in the proof of Theorem 1.

First consider any PHAP $P^*_f = (I^*, B^*, \succ^*, \rhd^*, \pi, a)$ where I^*, B^*, $\succ^*$, and $\rhd^*$ are defined as in the ex-post justified envy case. We moreover assume that π is a MAP with $\pi(a_t = \beta) = \frac{1}{2}$ for all $\beta \in B^*$. If $i_{\varphi(P)}(a_0) = i_2$ when $a_0 = \beta_1$, Case 1 of the ex-post justified envy case shows that $a_1 = \beta_1$ leads to ex-post justified envy. Therefore, the arrival sequence $a_0 = a_1 = \beta_1$, happening with probability $\frac{1}{4}$, induces a matching outcome in the support of $\mathcal{L}(P^*, \varphi)$ violating ex-post justified envy. Likewise, if $i_{\varphi(P)}(a_0) = i_2$ when $a_0 = \beta_1$, Case 2 of the ex-post justified envy case shows that $a_1 = \beta_2$ leads to ex-post justified envy. Therefore, the arrival sequence $a_0 = \beta_1$ and $a_1 = \beta_2$, happening with probability $\frac{1}{4}$, induces a matching outcome in the support of $\mathcal{L}(P^*_f, \varphi)$ violating ex-post justified envy.

Now consider any PHAP $P^*_e = (I^*, B^*, \succ^*, \rhd^*, \pi^*, a)$ where I^*, B^*, $\succ^*$, and $\rhd^*$ are defined as in the ex-post Pareto efficiency case. If $i_{\varphi(P)}(a_0) = i_1$ when $a_0 = \beta_1$, Case 1 of the ex-post Pareto efficiency case shows that $a_1 = \beta_3$ leads to an ex-post Pareto dominated matching outcome. Therefore, the arrival sequence $a_0 = \beta_1$ and $a_1 = \beta_3$, happening with probability $\frac{1}{16}$, induces a matching outcome in the support of $\mathcal{L}(P^*, \varphi)$ violating ex-post Pareto efficiency. If $i_{\varphi(P)}(a_0) = i_2$ when $a_0 = \beta_1$, Case 2 of the ex-post Pareto efficiency case shows that $a_1 = \emptyset$ and $a_2 = \beta_2$ lead to an ex-post Pareto dominated matching outcome. Therefore, the arrival sequence $a_0 = \beta_1$, $a_1 = \emptyset$, and $a_2 = \beta_2$, happening with probability $\frac{1}{64}$, induces a matching outcome in the support of $\mathcal{L}(P^*_e, \varphi)$ violating ex-post Pareto efficiency. □

A.6 Proof of Theorem 6

Proof of Theorem 6. Consider any given strategy-proof and non-wasteful mechanism φ, along with the following setup:

- **Applicants.** $I^* = \{i_1, i_2\}$.
- **Buildings.** $B^* = \{\beta_1, \beta_2\}$.
- **Preferences.** $\succ^*$ is a given preference profile such that

$$i_1 : (\beta_1, 0) \succ^*_{i_2} (\beta_2, 2) \succ^*_{i_2} i_2$$
$$i_2 : (\beta_2, 2) \succ^*_{i_1} (\beta_1, 0) \succ^*_{i_1} i_1$$

- **Priorities.** $\triangleright^*$ is any priority profile such that $i_2 \triangleright^* i_1 \triangleright^* \beta$ for all $\beta \in B^*$.
- **Arrival Distribution.** π^* is a MAP such that $\pi(a_t = \beta) = \frac{1}{2}$ for all $\beta \in B^*$.
- **Realized Arrivals.** $A^* = \{a \in A(\pi^*) : a_0 = \beta_1\}$.

Consider the PHAP subset

$$\mathcal{P}^* = \{P \in \mathcal{P} : I = I^*, B = B^*, \succ = \succ^*, \triangleright = \triangleright^*, \pi = \pi^*, a \in A^*\}.$$

Similarly to the proof of Theorem 1, non-wastefulness and the informational constraint imply that we can have only two cases: $i_{\varphi(P)}(a_0) = i_1$ for all $P \in \mathcal{P}^*$ or $i_{\varphi(P)}(a_0) = i_2$ for all $P \in \mathcal{P}^*$. Moreover, non-wastefulness implies that $i_{\varphi(P)}(a_1) \in I \setminus \{i_{\varphi(P)}(a_1)\}$, with $T_{\varphi(P)}(i_{\varphi(P)}(a_1)) \leq 1$. Notice that no uncertainty is realized for $t \in (0, 1)$, so $\tau(\langle \beta, r\rangle, T_{\varphi(P)}(i_{\varphi(P)}(a_1))) = \tau(\langle \beta, r\rangle, 0) - T_{\varphi(P)}(i_{\varphi(P)}(a_1))$ for any unit $\langle \beta, r\rangle$. Therefore, $T_{\varphi(P)}(i_{\varphi(P)}(a_1)) + \tau(\langle \beta, r\rangle, T_{\varphi(P)}(i_{\varphi(P)}(a_1))) = \tau(\langle \beta, r\rangle, 0)$.

Case 1: $i_{\varphi(P)}(a_0) = i_1$ and $i_{\varphi(P)}(a_1) = i_2$ for all $P \in \mathcal{P}^*$. This is particularly true for any $P^*_1 \in \mathcal{P}^*$ featuring $a_1 = \beta_1$. It follows that the matching outcome $\varphi(P^*_1)$ suffers from FS-interim justified envy. To see it, notice that $i_2 \triangleright_{\beta_1} i_1$ and $\tau(\langle \beta_1, 2\rangle, 0) = 2$, so

$$\overline{\varphi}_{P^*_1}(i_1, T_{\varphi(P^*_1)}(i_1)) = (\beta_1, 0) \succ^*_{i_2} (\beta_1, 2) = \overline{\varphi}_{P^*_1}(i_2, T_{\varphi(P^*_1)}(i_2)).$$

Case 2: $i_{\varphi(P)}(a_0) = i_2$ and $i_{\varphi(P)}(a_1) = i_1$ for all $P \in \mathcal{P}^*$. This is particularly true for any $P^*_2 \in \mathcal{P}^*$ featuring $a_1 = \beta_2$. It follows that the matching outcome $\varphi(P^*_2)$ is FS-interim Pareto dominated by the mechanism that swaps applicants' assignments. To see it, notice that $\tau(\langle \beta_2, 1\rangle, 0) = 2$, so

$$\overline{\varphi}_{P^*_2}(i_2, T_{\varphi(P^*_2)}(i_2)) = (\beta_1, 0) \succ^*_{i_1} (\beta_2, 2) = \overline{\varphi}_{P^*_2}(i_1, T_{\varphi(P^*_2)}(i_1))$$
$$\overline{\varphi}_{P^*_2}(i_1, T_{\varphi(P^*_2)}(i_1)) = (\beta_2, 2) \succ^*_{i_2} (\beta_1, 0) = \overline{\varphi}_{P^*_2}(i_2, T_{\varphi(P^*_2)}(i_2)). \qquad \square$$

A.7 Proof of Theorem 7

Proof of Theorem 7. *Part (i).* Demotion-CWL mechanisms.

- **Pareto efficiency and justified envy.** We divide the argument in two: $k \in \mathbb{N}$ and $k = \infty$ (FCFS).

  - Case 1, $k \in \mathbb{N}$. Consider the following PHAP with 2 buildings, $B = \{\beta, \beta'\}$, each with one unit; $k+2$ applicants, $I = \{i_j\}_{j=0}^{k+1}$; and $k+2$ discrete periods. Buildings have a common priority that ranks applicant i_j higher than applicant i_{j+1} for all $j = 0, \ldots, k$. Finally, units arrive deterministically: in periods 0 through k, a unit in building β' becomes available with certainty; in period $k+1$, a unit in building β becomes available with certainty. Applicant i_0 prefers the unit in building β that arrives in period $k+1$, and all other applicants prefer units that arrive earlier.

    Since applicant i_0 cannot attain any unit outside building β', they refuse no offers and immediately receive a unit in building β'. Applicants i_1 through i_{k+1} also do not refuse any offers, as they prefer earlier units.

    Applicant i_0 prefers the unit in building β' and therefore justifiably envies i_{k+1}. Moreover, matching applicant i_0 with the unit in building β and matching each of the remaining applicants with the room that arrives one period earlier would be a Pareto improvement.

  - Case 2, $k = \infty$. Consider two applicants, $I = \{i_1, i_2\}$ and three buildings $B = \{\beta, \beta', \beta''\}$. All buildings are acceptable to all applicants and vice versa. Buildings share a common priority $i_1 \rhd i_2$. We consider PHAP featuring arrivals only at periods periods $t \in \{0, 1, \ldots, 10\}$. The first arrival is $a_0 = \beta$ with certainty. Then, with probability $\frac{3}{4}$, $a_1 = \beta'$, and with probability $\frac{1}{4}$, $a_1 = \beta''$. Finally, regardless of a_1, with certainty $a_j = \beta$ for $j = 2, 3, \ldots, 8$, $a_9 = \beta'$, and $a_{10} = \beta''$.

    Applicant i_1's preferences are represented by the utility function $U(\beta, t) = u(\beta) - c_1 t$, with $u(\beta) = 19$, $u(\beta') = 30$, $u(\beta'') = 18$, and $c_1 = 2$. Applicant i_2's preferences are represented by the utility function $V(\beta, t) = v(\beta) - c_2 t$, with $v(\beta) = 25$, $v(\beta') = 39$, $v(\beta'') = 40$, and $c_2 = 2$. Applicants' preferences imply the following order, relevant for the proof:[3]

$$i_1 : (\beta', 1) \succ \tfrac{3}{4}(\beta', 1) + \tfrac{1}{4}(\beta'', 1) \succ (\beta, 0) \succ (\beta'', 1) \succ (\beta, 2) \succ (\beta', 9) \succ (\beta'', 10). \tag{13}$$

$$i_2 : (\beta'', 8) \succ (\beta, 0) \succ (\beta', 9) \succ (\beta'', 10). \tag{14}$$

    Consider the FCFS mechanism CWL-∞, where applicants can indefinitely reject units. We first analyze applicant i_1 by backwards induction and show that they

[3] In a slight abuse of notation, we write $\alpha(\beta, t) + (1-\alpha)(\beta', t')$ to denote "receiving building β at time t with probability α and building β' at time t' with probability $1-\alpha$."

always reject a_0, to wait for and accept a_1, regardless of its realization.

When i_1 remains unassigned at $t = 1$, they have the top priority among applicants so they are offered a_1, regardless of the unit realization. Suppose that i_1 is offered a_1 at $t = 1$. By accepting the unit, they attain either matching outcome $(\beta', 1)$ or $(\beta'', 1)$. If they reject the unit and waits, they keep the top priority and can secure one matching outcome among (β, s) for $s = 2, \ldots, 8$, $(\beta', 9)$, and $(\beta'', 10)$. Equation (13) and the costly-waiting property imply that i_1 maximizes their preferences by accepting a_1, regardless of its realization.

Therefore, at $t = 0$, when i_1 is offered a_0 as the top-priority applicant, they know that rejecting the unit ultimately leads to accepting a_1. By accepting a_0, applicant i_1 secures matching outcome $(\beta, 0)$. By rejecting a_0 and waiting for a_1, applicant i_1 secures matching outcome $(\beta', 1)$ with probability $\frac{3}{4}$ and matching outcome $(\beta'', 1)$ with probability $\frac{3}{4}$. From Equation (13), $\frac{3}{4}(\beta', 1) + \frac{1}{4}(\beta'', 1) \succ_{i_1} (\beta, 0)$ so applicant i_1 rejects a_0 and waits for a_1 instead.

Now, when a_0 is rejected by i_1, the FCFS mechanism offers the unit to i_2. We show that the fact that i_1 will accept a_1 makes i_2 accept a_0. On the one hand, if i_2 accepts a_0, they secure matching outcome $(\beta, 0)$. Meanwhile, as i_1 accepts a_1, if i_2 rejects a_0, they can secure one matching outcome among (β, s) for $s = 2, \ldots, 8$, $(\beta', 9)$, and $(\beta'', 10)$. Equation (14) and the costly-waiting property imply that i_2 maximizes their preferences by accepting a_0.

We conclude that, regardless of the realization of a_1, the FCFS mechanism assigns a_0 to applicant i_2 and a_1 to applicant i_1. Consider the PHAP P featuring $a_1 = \beta''$. Then, the FCFS mechanism, denoted φ^F, leads to outcome $\varphi^F_P(i_1) = (\langle \beta'', 1 \rangle, 1)$ and $\varphi^F_P(i_2) = (\langle \beta, 1 \rangle, 0)$, with assignment times $T_{\varphi^F(P)}(i_1) = 1$ and $T_{\varphi^F(P)}(i_2) = 0$. Note that Equation (13) implies that

$$\overline{\varphi}^F_P(i_2, T_{\varphi^F(P)}(i_1)) = (\langle \beta, 1 \rangle, 0) \succ_{i_1} (\langle \beta'', 1 \rangle, 1) = \overline{\varphi}^F_P(i_1, T_{\varphi^F(P)}(i_1)). \tag{15}$$

As $i_i \triangleright_{\beta''} i_2$, we conclude that φ^{CBWL} does not eliminate justified envy from an interim perspective. Moreover, note that $\tau_{\varphi^F(P)}(\langle \beta'', 1 \rangle, 0) = \frac{3}{4}10 + \frac{1}{4} = 7.75$. Therefore, the first inequality in Equation (14) and costly waiting imply that

$$\overline{\varphi}^F_P(i_1, T_{\varphi^F(P)}(i_2)) = (\langle \beta'', 1 \rangle, 7.75) \succ_{i_2} (\langle \beta, 1 \rangle, 0) = \overline{\varphi}^F_P(i_2, T_{\varphi^F(P)}(i_2)). \tag{16}$$

It follows that $\varphi^F(P)$ is interim Pareto dominated by $\phi(P)$, for mechanism ϕ defined as the mechanism that maintains applicants' assignment times under φ^F

unaffected but swaps their assignments.

- **Non-wastefulness.** We divide the proof into two cases.

  – *Case 1,* $k \geq 2$. Consider any demotion-CWL-2 mechanism φ^2 and the following PHAP: $B = \{\beta, \beta'\}$, $I = \{i_1, i_2\}$, deterministic arrivals only at $t = 0, 1, 2$, $a_0 = \beta$ and $a_1 = a_2 = \beta'$, and $(\beta', 2) \succ_i (\beta, 0)$ for all $i \in I$. Finally, both buildings are acceptable to both applicants and vice versa. Suppose $i_1 \triangleright i_2$ in both buildings so the mechanism chooses i_1 before i_2.

    As argued in the proof of Theorem 4, under deterministic arrivals, $t_{\varphi(P)}(i) = T_{\varphi(P)}(i) + \tau_{\varphi(P)}(\mu_{\varphi(P)}(i), T_{\varphi(P)}(i))$, and so our interim notions coincide with our ex-post notions.

    First, the mechanism chooses i_1, who considers the matching outcome $(\beta, 0)$. Alternatively, if i_1 declines and waits until $t = 1$, they can secure the matching outcome $(\beta', 1) \succ_{i_1} (\beta, 0)$, so i_1 declines and waits. Then, the mechanism offers matching outcome $(\beta, 0)$ to i_2. If they decline and wait, in $t = 1$ applicant i_1 accepts $\langle \beta', 1 \rangle$, so in $t = 2$, the mechanism offers $\langle \beta', 2 \rangle$ to i_2. Therefore, in $t = 0$, by declining and waiting, then accepting an offer in $t = 2$, they secure matching outcome $(\beta', 2) \succ_{i_2} (\beta, 0)$. Therefore, in $t = 0$, i_2 declines and waits for a future unit.

    As no additional applicants exist, a_0 goes unassigned ($i_{\varphi^2(P)}(a_t) = a_t$), leading to a violation of non-wastefulness, given that both applicants are acceptable to and find acceptable β.

  – *Case 2,* $k = 1$. Consider the demotion-CWL-1. If applicants decline a unit, they remain on the waiting list after being placed at the bottom (d_i increases by one). Consider a PHAP with only one applicant, i. Suppose that $a_0 = \beta$ and $a_1 = \beta'$ with certainty. i is acceptable both at β and β' and $(\beta', 1) \succ_i (\beta, 0) \succ_i i$. Given that i is the only applicant, they know that by rejecting a_0, they will be offered a_1 with certainty. Given i's preferences, they decline a_0 and are matched to a_1. As no applicants remain, a_0 goes unmatched, even though $i \triangleright_\beta \beta$ and $\beta \succ_i i$. Hence, non-wastefulness fails.

Part (ii). Ultimatum-CWL mechanisms.

- **Pareto efficiency and justified envy.** The example in the proof of Pareto efficiency and justified envy for demotion-CWL mechanisms applies without change for ultimatum-CWL mechanisms.

- **Non-wastefulness.** We divide the proof into two cases.
  - *Case 1, $k \geq 2$.* The example in the proof of non-wastefulness when $k \geq 2$ for demotion-CWL mechanisms applies without change for ultimatum-CWL mechanisms.
  - *Case 2, $k = 1$.* Consider the ultimatum-CWL-1 mechanism. If applicants decline a unit, they are deleted from the waiting list and go unmatched. Suppose that i is the selected applicant for a given arrival $a_t = \beta$, which means that $i \rhd_{a_t} a_t$. If i declines a_t, they go unmatched because they are deleted from the waiting list. Therefore, whenever $a_t \succ_i i$, the applicant always maximizes their preferences by accepting the unit; consequently, the conditions for non-wastefulness hold for i. As applicant i is arbitrary, non-wastefulness attains.

Part (iii). SBWL Mechanisms.

- **Pareto efficiency and justified envy.** We divide the argument in two: $k = 1$ and $k > 1$.
  - *Case 1, $k = 1$.* Consider the SBWL-1 mechanism and the following PHAP. There are only two applicants, i_1 and i_2. There are also two buildings, β_1 and β_2. Both buildings share the same priority $i_1 \rhd i_2 \rhd \emptyset$. Both applicants find both buildings acceptable. Applicant i_1's preferences can be represented by the utility function $V(\beta, t) = v_\beta - c_1 t$, where $v_{\beta_1} = 3$, $v_{\beta_2} = 1$, and $c_1 = 1$. Applicant i_2's preferences can be represented by the utility function $U(\beta, t) = u_\beta - c_2 t$, where $u_{\beta_1} = 4$, $u_{\beta_2} = 3$, and $c_2 = 3$. Finally, units only arrive during integer periods in the following way. On the one hand, with probability $\frac{1}{6}$, the arrival process is deterministic with arrivals $a_0 = \emptyset$, $a_{2n-1} = \beta_2$ and $a_{2n} = \beta_1$, for $n \in \mathbb{N}$. On the other hand, with probability $\frac{5}{6}$, the arrival process is deterministic with arrivals $a_0 = a_6 = \beta_2$, $a_1 = a_2 = a_3 = a_5 = a_7 = \emptyset$, and $a_4 = a_8 = \beta_2$.

    Before the mechanism is executed and units arrive, i_1 and i_2 play a simultaneous game of incomplete information to decide whether to join Q_{β_1}, Q_{β_2} or no list. The Bayesian Nash Equilibrium (BNE) of such a game determines applicants' queues going into the mechanism.

    Two observations simplify the analysis. First, given applicants' priorities, i_1 claims the top position at any waiting list they join, whereas i_2 claims the top position on a waiting list only when i_1 is not on that list. Second, not joining any waiting list is a strictly dominated strategy for both applicants, as it amounts to remaining unmatched when both buildings are acceptable to them, and two units on each building are set to arrive with certainty.

If i_1 joins Q_{β_2}, they always receive $\langle \beta_2, 1 \rangle$. With probability $\frac{1}{6}$, the unit is delivered at time $t = 1$ and, with probability $\frac{5}{6}$, it is delivered at $t = 0$. Therefore, the applicant's expected utility of joining Q_{β_2} is $\frac{5}{6}$. Meanwhile, if applicant i_1 joins Q_{β_1}, they always receive $\langle \beta_1, 1 \rangle$. With probability $\frac{1}{6}$, the unit is delivered at time $t = 2$ and, with probability $\frac{5}{6}$, it is delivered at $t = 4$. Therefore, the applicant's expected utility of joining Q_{β_1} is $-\frac{2}{3}$. Regardless of i_2's choice, applicant i_1 strictly prefers to join Q_{β_2}.

Suppose now that applicant i_1 joins Q_{β_2}. If i_2 joins Q_{β_2}, they always receive $\langle \beta_2, 2 \rangle$. With probability $\frac{1}{6}$, the unit is delivered at time $t = 3$ and, with probability $\frac{5}{6}$, it is delivered at $t = 6$. Therefore, the applicant's expected utility of joining Q_{β_2} is $-\frac{27}{2}$. Meanwhile, if applicant i_1 joins Q_{β_1}, they always receive $\langle \beta_1, 1 \rangle$. With probability $\frac{1}{6}$, the unit is delivered at time $t = 2$ and, with probability $\frac{5}{6}$, it is delivered at $t = 4$. Therefore, the applicant's expected utility of joining Q_{β_1} is -7. When i_2 joins Q_{β_2}, applicant i_1 strictly prefers to join Q_{β_1}.

We conclude that the only BNE of the game consists of i_1 joining Q_{β_2} and i_2 joining Q_{β_1}. Suppose that the realized sequence of arrivals is $a_0 = \emptyset$, $a_{2n-1} = \beta_2$, and $a_{2n} = \beta_1$ for all $n \in \mathbb{N}$. We have that $T_{\varphi(P)}(i_1) = 1$ and $T_{\varphi(P)}(i_2) = 2$. Moreover, after $t = 0$, both applicants can distinguish the state of the world; thus, the rest of the arrivals considered are deterministic. As argued in the proof of Theorem 4, under deterministic arrivals, $t_{\varphi(P)}(i) = T_{\varphi(P)}(i) + \tau_{\varphi(P)}(\mu_{\varphi(P)}(i), T_{\varphi(P)}(i))$, and so our interim notions coincide with our ex-post notions.

Note that $\overline{\varphi}_P(i_2) = (\langle \beta_1, 1 \rangle, 2) \succ_{i_1} (\langle \beta_2, 1 \rangle, 1) = \overline{\varphi}_P(i_1)$. Moreover, $i_1 \rhd_{\beta_1} i_2$, so the SBWL-1 fails to eliminate interim justified envy. Moreover, $\overline{\varphi}_P(i_1) = (\langle \beta_2, 1 \rangle, 1) \succ_{i_2} (\langle \beta_1, 1 \rangle, 2) = \overline{\varphi}_P(i_2)$. Therefore, the mechanism that swaps assignments between applicant i_1 and i_2 interim Pareto dominates the SBWL-1 mechanism.

- *Case 2, $k > 1$.* Consider the same PHAP from Case 1. Now, the simultaneous game of incomplete information involves an additional strategy for each player: joining both Q_{β_1} and Q_{β_2}. Note that applicant i_1 is the top priority in both buildings. Given that in both states of the world β_2 is the first arriving unit, joining both Q_{β_1} and Q_{β_2} is outcome-equivalent to joining Q_{β_2} only, leading to i_2 receiving $\langle \beta_2, 1 \rangle$. Given this fact, and the fact that $\langle \beta_1, 1 \rangle$ is the second arriving unit in both states of the world, joining both Q_{β_1} and Q_{β_2} is outcome-equivalent to joining Q_{β_1} only, leading to i_1 receiving $\langle \beta_1, 1 \rangle$. We conclude that the violations to interim Pareto efficiency and elimination of justified envy from Case 1 remain valid in any BNE.[4]

- **Non-wastefulness.** Consider any SBWL-k mechanism and the following PHAP. There is only one applicant, i, and two buildings, β_1 and β_2. Applicant i has the following preference: $(\beta_1, 1) \succ_i (\beta_2, 0) \succ_i i$. Moreover, applicant i is acceptable at both buildings. Finally, the arrival process is deterministic, with $a_0 = \beta_2$ and $a_1 = \beta_1$. In this setup, if applicant i chooses Q_{β_2} or both Q_{β_1} and Q_{β_2}, they are assigned to $\langle \beta_2, 1 \rangle$ with delivery time $t = 0$. Meanwhile, if applicant i chooses Q_{β_1} only, they are assigned to $\langle \beta_2, 1 \rangle$ with delivery time $t = 1$. Therefore, i joins Q_{β_1}. This implies, however, that $i_{\varphi(P)}(a_0) = a_0$. Therefore, as i is available to match with a_0 at $t = 0$, $i \triangleright_{\beta_2} \beta_2$, and $\beta_2 \succ_i i$, we conclude that the mechanism violates non-wastefulness. □

[4]Given the outcome equivalence between choosing Case 1's queue or both queues, applicant i_1 is indifferent between joining Q_{β_2} and joining both waiting lists. Likewise, applicant i_2 is indifferent between joining Q_{β_1} and joining both waiting lists. Therefore, any strategy profile $(s_{i_1}, s_{i_2}) \in \{\{Q_{\beta_2}\}, \{Q_{\beta_1}, Q_{\beta_2}\}\} \times \{\{Q_{\beta_1}\}, \{Q_{\beta_1}, Q_{\beta_2}\}\}$ is BNE leading to the outcome of Case 1.

B Existing Waiting List Mechanisms

B.1 The Ultimatum-CWL-k Mechanism

Step 0: Initialization

i. Create an unordered waiting list L with all applicants.

ii. Create an offer rejection counter $R_i = 0$ for every applicant $i \in L$.

Step 1: Arrival of units and creation of rejections list

i. Terminate the algorithm if $L = \emptyset$, if no units are left to arrive with strictly positive probability, or if every unit left to arrive with positive probability finds every applicant left in L unacceptable.

ii. Otherwise, move forward to the next period with a unit arrival. Denote the arriving unit by $\langle \beta_A, r_A \rangle$.

iii. Create an unordered empty list $L_{\langle \beta_A, r_A \rangle}$ and proceed to Step 2.

Step 2: Process the waiting list L

i. If $L = L_{\langle \beta_A, r_A \rangle}$, discard $\langle \beta_A, r_A \rangle$ and return to Step 1. Otherwise, determine $i^* \in L \setminus L_{\langle \beta_A, r_A \rangle}$ with the highest priority at β_A.

ii. If i^* is not acceptable at β_A, discard $\langle \beta_A, r_A \rangle$ and return to Step 1. Otherwise, offer $\langle \beta_A, r_A \rangle$ to i^*.

iii. If i^* accepts $\langle \beta_A, r_A \rangle$, delete i^* from L and return to Step 1. Otherwise, proceed to Step 3.

Step 3: Update $L_{\langle \beta_A, r_A \rangle}$ and punish offer rejection

i. Increase R_{i^*} by one.

ii. If $R_{i^*} = k$, delete i^* from L and return to Step 2. Otherwise, add i^* to $L_{\langle \beta_A, r_A \rangle}$ and return to Step 2.

B.2 The Demotion-CWL-k Mechanism

Step 0: Initialization

i. Create an unordered waiting list L with all applicants.

ii. For every applicant $i \in L$, create an offer rejection counter $R_i = 0$ and a participation counter $d_i = 1$.

Step 1: Arrival of units and creation of rejections list

i. Terminate the algorithm if no units are left to arrive with strictly positive probability or if every unit left to arrive with positive probability finds every applicant left in L unacceptable.

ii. Otherwise, move forward to the next period with a unit arrival. Denote the arriving unit by $\langle \beta_A, r_A \rangle$.

iii. Create unordered empty lists $L_{\langle \beta_A, r_A \rangle}$ and $D_{\langle \beta_A, r_A \rangle}$. Proceed to Step 2.

Step 2: Process the waiting list L

i. Discard $\langle \beta_A, r_A \rangle$ and return to Step 1 if $L = L_{\langle \beta_A, r_A \rangle}$ or the applicant in $L \setminus L_{\langle \beta_A, r_A \rangle}$ with the highest priority at β_A is unacceptable.

ii. Otherwise, offer $\langle \beta_A, r_A \rangle$ to the applicant $i^* \in L \setminus L_{\langle \beta_A, r_A \rangle}$ determined using the following rule: i^* is the β_A-acceptable applicant in $L \setminus L_{\langle \beta_A, r_A \rangle}$ with the smallest participation counter d_i. If more than one such applicant exists, i^* has the highest priority among them at β_A.

iii. If i^* accepts $\langle \beta_A, r_A \rangle$, delete i^* from L and return to Step 1. Otherwise, proceed to Step 3.

Step 3: Update $L_{\langle \beta_A, r_A \rangle}$ and punish offer rejection

i. Increase R_{i^*} by one and add i^* to $L_{\langle \beta_A, r_A \rangle}$.

ii. If $R_{i^*} = k$, increase d_{i^*}, restart $R_{i^*} = 0$, and return to Step 2.

iii. Otherwise, return to Step 2 without additional adjustments.

B.3 The SBWL-k Mechanism

Step 0: Initialization

i. Create an empty waiting list Q_β for each building $\beta \in B$.

ii. Collect a list of up to k waiting lists to join from each applicant.

iii. Place applicants in their desired waiting lists and order each list according to the corresponding building's priority orderings $\triangleright$. Proceed to Step 1.

Step 1: Arrival of units

i. Terminate the algorithm if $Q_\beta = \emptyset$ for all $\beta \in B$ of if no units are left to arrive with strictly positive probability.

ii. Otherwise, move forward to the next period with a unit arrival. Denote the arriving unit by $\langle \beta_A, r_A \rangle$. Proceed to Step 2.

Step 2: Process queue for building Q_{β_A}

i. If $Q_{\beta_A} = \emptyset$, discard $\langle \beta_A, r_A \rangle$ and return to Step 1.

ii. Otherwise, assign $\langle \beta_A, r_A \rangle$ to the applicant atop of Q_{β_A}, delete such an applicant from Q_{β_A}, and return to Step 1.

References

Murra-Anton, Zeky, and Neil Thakral. 2024. "The Public Housing Allocation Problem." *Mimeo.*

C Appendix Tables

Appendix Table 1: Allocation mechanisms in largest PHAs

Geographic area	Mechanism	Population	Number of units
New York City, NY	SBWL-1	8,335,897	160,741
Chicago, IL	CWL-1, SBWL-1	2,665,039	15,319
Philadelphia, PA	CWL-1, SBWL-5	1,567,258	12,799
Boston, MA	SBWL-∞	650,706	9,107
Washington, D.C.	CWL-2, SBWL-3	671,803	7,922
Miami, FL	CWL-1	449,514	6,929
Baltimore, MD	CWL-1, SBWL-3	569,931	6,903
Newark, NJ	CWL-1, SBWL-3	305,344	6,457
Los Angeles, CA	CWL-2 (or 3)	3,822,238	6,386
Cuyahoga County, OH	SBWL-∞	1,236,994	6,120
San Antonio, TX	SBWL-∞	1,472,909	6,049
Minneapolis, MN	CWL-1, CWL-3	425,096	5,358
Seattle, WA	SBWL-2	749,256	5,229
Hawaii	SBWL-1	1,439,399	4,719
Birmingham, AL		196,910	4,688
Cincinnati, OH	SBWL-∞	309,513	4,552
Akron, OH	CWL-1	188,509	4,307
Buffalo, NY	SBWL-1	276,486	4,202
Louisville, KY	CWL-2	624,444	3,739
Richmond, VA	CWL-2	229,395	3,499
Detroit, MI	SBWL-∞	620,376	3,394
Dallas, TX	SBWL-∞	1,299,544	3,160
Pittsburgh, PA	CWL-1, SBWL-3	302,898	3,154
Denver, CO	CWL-2	713,252	2,967
Los Angeles County, CA	CWL-1	9,719,765	2,953
Oklahoma City, OK	SBWL-∞	694,800	2,913
St. Louis, MO	SBWL-∞	286,578	2,807
Allegheny County, PA	SBWL-∞	1,232,605	2,780
Mobile, AL	CWL-2	183,289	2,653
Houston, TX	SBWL-3	2,302,878	2,609
Providence, RI	CWL-1	189,563	2,605
Lucas County, OH	CWL-2	426,719	2,586
Omaha, NE	SBWL-∞	485,153	2,545
King County, WA	SBWL-2	2,265,311	2,447
Worcester, MA	CWL-3	205,319	2,438

Note: This table presents information about the housing allocation mechanisms used in the 35 largest Public Housing Authorities in the United States. Population figures come from the Census Bureau (July 2022), and the number of public housing units in each area comes from the Picture of Subsidized Households (FY 2023) dataset from the Office of Policy Development and Research at the Department of Housing and Urban Development. Information about the allocation mechanisms comes from each housing authority's Admissions and Continued Occupancy Policy document (not available online for Birmingham).